~~#4 Sequence — A succession of numbers~~

#4.1 Sequence :- Sequence is a succession of terms according to definite rule.

#4.2 Series :- A series is defined as the sum of terms of a sequence.

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$$

4.3. General properties of Series :-

1. The replacement, inclusion or omission of a finite number of terms does not change the convergence of series.
2. When each elements of the series is multiplied by a fixed number other than zero then this series remain convergent, divergent or oscillatory.
3. Every finite series is convergent
4. A series of positive terms either converges or diverges to $+\infty$.

Finite and Alternating Series :-

- A finite series is a series which the number of terms in a series is finite.

$$u_1 + u_2 + u_3 + \cdots + u_n \text{ or } \sum_{r=1}^{n} u_r$$

- The Alternating series is a series where the terms are alternately positive and negative.

$$\sum (-1)^n . u_n = u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^n . u_n + \cdots,$$

where $u_n > 0, \forall n$

Infinite Series :- If $\langle u_n \rangle$ is a sequence of real numbers, then the expression $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is called an infinite series.

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The infinite Series
$u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$ or $\sum u_n$.

Ex. ① $\sum_{n=1}^{\infty} \frac{1}{n}$ ② $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ ③ $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n+1}}$ ④ $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)}$

Infinite Series of positive terms:—

The series $\sum_{n=1}^{\infty} u_n$ is called a series of positive terms if all the terms of series $\sum u_n = u_1 + u_2 + \cdots + u_n + \cdots$ are positive i.e. if $u_n > 0$, $\forall n$.

Let $\sum u_n$ be an infinite series of positive terms and sequence $\{S_n\}$ of its partial sums, so that

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n \geq 0, \quad \forall n$$

$$\therefore S_n - S_{n-1} = u_n \geq 0 \Rightarrow S_n \geq S_{n-1}, \quad \forall n > 1$$

Theorem:- A necessary and sufficient condition for the convergence of a positive terms series $\sum_{n=1}^{\infty} u_n$ is that the sequence $\langle S_n \rangle$ of the partial sums of the series defined by $S_n = u_1 + u_2 + \cdots + u_n$ is bounded above.

Proof:- ① The necessary condition:-

As $u_n \geq 0$, $n \in \mathbb{N}$, the sequence $\langle S_n \rangle$ is monotonically increasing.

$\therefore$ the series is convergent.

(ii) The Sufficient Condition:-

The sequence $\langle S_n \rangle$, which is monotonically increasing to be convergent is that it is bounded.

A positive term series is divergent if and only if the sequence of partial sums is not bounded above.

Theorem :- If a series of positive terms $\sum_{n=1}^{\infty} u_n$ is convergent, then prove that $\lim_{n\to\infty} u_n = 0$. Show that the converse is not always true.

Proof :- (i) The necessary condition :-

Let the series $\sum u_n$ is convergent.

Then, to prove $\lim_{n\to\infty} u_n = 0$.

Let S_n be the sums of n terms of series $\sum u_n$

Then, $S_n = u_1 + u_2 + \cdots + u_n$

$\therefore S_{n-1} = u_1 + u_2 + \cdots + u_{n-1}$

$\therefore u_n = S_n - S_{n-1}$

Hence, u_n is convergent

$\therefore S_n$ and S_{n-1} both tend to same finite value. Say S when $n \to \infty$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - S_{n-1} = S - S = 0$ Proved

(ii) The sufficient condition :-

The converse of this theorem is not true. It is shown by given example.

Consider the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$$

Here, $u_n = \frac{1}{\sqrt{n}}$ then $\lim_{n\to\infty} u_n = 0$

Given that $S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}$

$\therefore S_n$ tends to infinity when $n \to \infty$ $\sqrt{\infty}$

Hence, the series is divergent.

Proved

Example: show that the Series $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\dots$ is not Convergent

$$\therefore \quad u_n = \frac{n}{n+1}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{n/n}{n/n + 1/n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1 \neq 0$$

Since, $\lim_{n\to\infty} u_n \neq 0$, the series is not Convergent.

~~Example: Solve~~

Convergence or Divergence of an infinite Series:—

Consider an infinite series $\sum_{n=1}^{\infty} u_n$ and

S_n = nth partial Sum of infinite Series

① Convergent Series:- The Series $\sum_{n=1}^{\infty} u_n$ Converges if the Sequence $<S_n>$ of partial Sums Converges.

i.e. If $\lim_{n\to\infty} S_n$ = finite then $\sum_{n=1}^{\infty} u_n$ is also Convergent.

② Divergent Series:— The series $\sum_{n=1}^{\infty} u_n$ diverges if Sequence $<S_n>$ of partial sums diverges.

i.e. If $\lim_{n\to\infty} S_n = +\infty$ or $-\infty$ then $\sum_{n=1}^{\infty} u_n$ is also divergent.

③ Oscillating Series:- The Series $\sum_{n=1}^{\infty} u_n$ is oscillates finitely if the Sequence $<S_n>$ of partial Sums oscillates finitely.

i.e. If $<S_n>$ is bounded and neither Converges nor diverges.

~~④ If~~

Note:- ~~If $<S_n>$ of partial Sums oscillates infinitely~~

If $<S_n>$ is unbounded and neither Converges nor diverges.